A Transformation Approach to Nonlinear Process Control

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This paper focusses on the approximate modeling and control of nonlinear processes. A polynomial expansion method helps develop successively more accurate models valid for the whole operating region and also provides a means to quantify the uncertainty associated with the approximation. A stability analysis can be performed in a straightforward fashion to identify stable operating regions as well as stable directions for the output variable moves. This permits the testing of linear and nonlinear compensators and the practicality of the method is demonstrated with a case study of heat-exchanger control.

Introduction

One of the outstanding problems in chemical process control is understanding the control related issues in nonlinear systems. While the nonlinear behavior of many chemical processes ranging from CSTRs to high-purity distillation columns has long been recognized, systematic ways of dealing directly with the nonlinear phenomena were lacking. Recently, linearizing transformation techniques pioneered by Hunt et al. (1983) attracted attention from the chemical engineering community (Hoo and Kantor, 1985; Kravaris and Chung, 1987; Calvet and Arkun, 1988) and led to significant contributions in the areas of disturbance rejection, observer design and enjoyed many applications to chemical process systems. This is a developing area of research where certain restrictions on systems still exist and computational problems may be an issue.

It is possible to avoid some of the restrictions of these nonlinear control methods by resorting to the use of approximate models. It should be noted that every model is subject to some degree of uncertainty and if one has the means to quantify this uncertainty, the use of approximate models is greatly justified. One of the contributions of approximate models as introduced by Khambanonda et al. (1990) is their ability to yield a viable stability test for nonlinear feedback systems. These models are based on polynomial expansions and are valid within the operating region of the process as opposed to a region near the point of expansion. This approximate model representation has significant potential in designing and testing *linear* and *nonlinear* controllers for nonlinear processes. In this study, we have three goals:

1. To discuss the approximations of nonlinear systems through polynomial transformations and to elaborate on the trade-offs involved in reaching a reasonable order model.
2. To create a setting in which linear P/PI-type controllers can be tuned and tested for nonlinear systems.
3. To assess the utility of this class of models in the design of nonlinear feedback controllers.

These goals will help establish our contention that polynomial based approximate models have a significant potential in tackling nonlinear control problems. Another point that should not be overlooked is the opportunity offered by our methodology to design simple linear controllers for nonlinear systems. Furthermore, a second-order nonlinear compensator can be designed to deal more effectively with system nonlinearities. The latter controller is very simple to compute and implement, primarily due to the polynomial nature of the process model used.

The article is structured as follows: we will first present the underlying principles of the polynomial expansions and elaborate on the related problems. Next, the controller design issue will be tackled by introducing linear and nonlinear controllers within a stability test setting. Implications of these developments will be investigated with a case study to shed further light on the use of the method.

Nonlinear System Transformation

In this section, we will briefly review some of the concepts previously introduced by Khambanonda et al. (1990) to set the background. The nonlinear system is assumed to be given in the control affine form by the vector equation:

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$$\dot{z} = f(z) + g(z)\mu \tag{1}$$

with

$$f(0) = 0$$

$$g(0) \neq 0$$

where $z \epsilon Z \subset \boldsymbol{R}^n$ and $\mu \epsilon U \subset \boldsymbol{R}^m$. Here, Z and U are bounded sets defining a "desired operating region" for the process, and they include the origin. The time argument of the functions is omitted for simplicity. Open-loop stable plants are also assumed in the sequel. Furthermore, the approach and the results presented heretofore are valid as long as $z(t)$ and $\mu(t)$ remain bounded within the operating region.

If we expand $f(z)$ and $g(z)$ in homogeneous powers of (z, μ), and define $f^{(2)}$ and $g^{(1)}$ as an n-dimensional vector field and an $n \times n$ matrix, respectively, then Eq. 1 can be expressed as:

$$\dot{z} = \Gamma z + \Psi \mu + f^{(2)}(z) + g^{(1)}(z)\mu + (\text{HOT}) \tag{2}$$

where

$$\Gamma = \{\gamma_{ij}\};\ \Psi = \{\psi_{ij}\};\ f_i^{(2)}(z) = \beta_{ijk} z_j z_k;\ g_i^{(1)}(z) = \varphi_{ijl} z_j \tag{3}$$

with $i, j = 1, \ldots n$; $k = j, \ldots n$ and $l = 1, \ldots m$. In this polnomial expansion, the vector of higher order terms (HOT) can be expressed in the following fashion:

$$(\text{HOT}) = D(z,\mu) \cdot z \tag{4}$$

with $D(z,\mu) = \text{diag}[d_1(z,\mu),\ d_2(z,\mu),\ \ldots,\ d_n(z,\mu)]$. The parameter $d_i(z,\mu)$ denotes a scalar uncertainty within the bounds:

$$d_i^1 \leq d_i(z,\mu) \leq d_i^u \tag{5}$$

These bounds can be computed as illustrated by Khambanonda et al. (1990) and the uncertainty representation can be used exclusively to quantify modeling errors and possible unmeasured disturbances somewhat similar to the formulation of Kantor and Keenan (1987). Equation 2 reduces the expansion into a second-order system given by:

$$\dot{z} = [\Gamma + D(z,\mu)]z + \Psi \mu + f^{(2)}(z) + g^{(1)}(z)\mu \tag{6}$$

Hence the expansion transforms Eq. 1 into a second-order polynomial while still retaining the effect of HOT.

Remark 1. It has to be noted that higher-order expansions are also possible and can be used to attenuate the magnitude of the uncertainty. This will be discussed shortly.

Remark 2. This expansion leads to an identification strategy to compute the coefficient matrices in Eq. 6. The choice of the identification strategy is also critical and the definition of the "desired operating region" will help establish the boundaries of the search region.

A Least-square approach to parameter determination

To determine the coefficients in Eq. 3, one can choose $d_i(z, \mu)$ to vanish at various steady-state locations within the "desired operating region". This happens if and only if

$$f(z^{s,j}) = \Gamma z^{s,j} + f^{(2)}(z^{s,j}) \tag{7a}$$

$$g(z^{s,j}) = \Psi + g^{(1)}(z^{s,j}) \tag{7b}$$

with $j = 1, 2, \ldots p$. Here $z^{s,j}$ is the value of the state vector at the jth steady-state point and p is the number of steady states selected. The number of parameters to be determined for the second order approximation is $N_p = n[2n + nm + m + (n-1) + (n-2) + \ldots + 1]$. It must be noted that the number of steady states selected should be at least equal to the number of coefficients to be calculated. Generally p will be selected sufficiently large to minimize $d_i(z,\mu)$. When p is chosen to be less than the number of coefficients, the solution of Eq. 7 is not unique, thus resulting in large uncertainty bounds. If $p = 2n + (n-1) + (n-2) + \ldots + 1$ and $p = m + nm$ respectively, Eq. 7 can be solved by direct inversion to obtain a unique solution. When p is selected to exceed the number of coefficients, we have redundancy and one is forced to seek for an approximate solution. One alternative is to find $f^{(2)}$ and $g^{(1)}$ which are closest, in a least-square sense, to $f(x)$ and $g(x)$. If this is the case, redundancy can be exploited and the error can be distributed within the operating region according to some predetermined weighting criterion.

$$\min_{\Gamma, f^{(2)}} \sum_{i=1}^{p} \| w_i [f(z^{s,i}) - \Gamma z^{s,i} - f^{(2)}(z^{s,i})] \|_2 \tag{8a}$$

$$\min_{\Psi, g^{(1)}} \sum_{i=1}^{p} \| v_i [g(z^{s,i}) - \Psi - g^{(1)}(z^{s,i})] \|_2 \tag{8b}$$

where w_i and v_i are the appropriate weights. This approach is fundamentally different from typical "Taylor expansion" type approaches since it accounts for the whole "desired operation region" rather than around a single point of expansion.

If one wishes to seek for an exact solution at the point (z^0, μ^0), this can be realized by choosing and assigning a relatively large weight to this nominal operating point. Two important factors that must be taken into consideration are:

1. Number of steady states to be selected (p).
2. Locations of the steady states to be selected.

The value of p and the locations of the selected steady states can be obtained following a procedure similar to the one employed by Gatica et al. (1987) for the design of experiments for parameter estimation. In that work, the selection is based on the comparison of the trace of the covariance matrix of the estimation error. A simple selection would be an evenly spaced grid points covering the desired operating region.

The case study will demonstrate the effect of using the least-square approach on the approximate model and the resulting stability predictions.

Higher-order polynomial expansions

As mentioned previously, one can develop higher-order approximations to improve the model accuracy. While higher-order models possess large number of parameters to be determined, the additional computational effort is not prohibitive. Naturally, there is a limit to the order of approximation at which no significant improvement is obtained by further increases. Higher-order models, on the other hand, will introduce undesirable complexities in the analysis of the system and

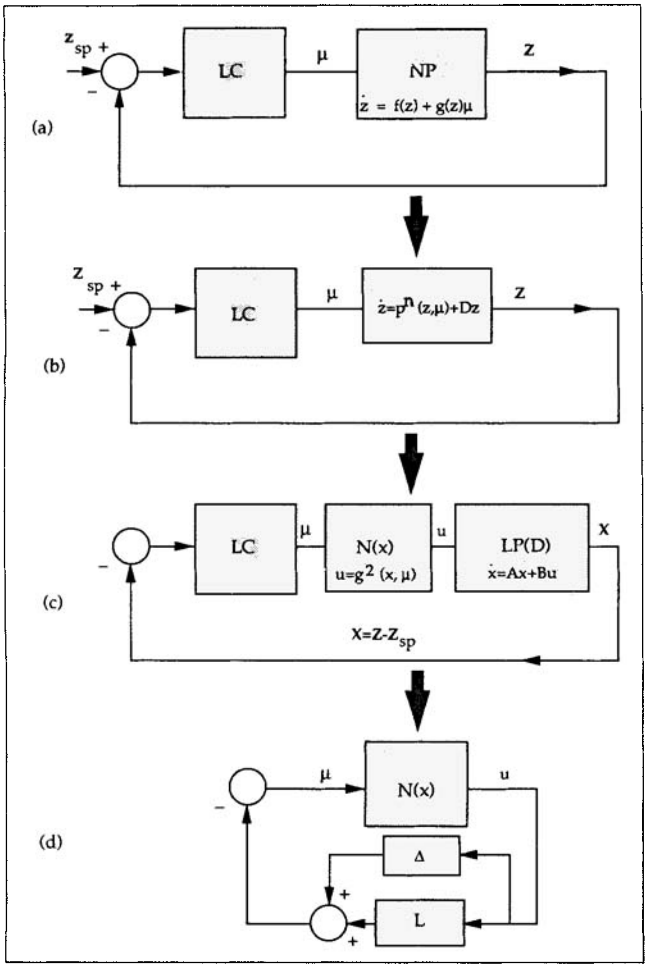

Figure 1. Evolution of the nonlinear transformation with arbitrary linear controller.

design of the control system. One must realize that order determination is system-dependent, but second- and third-order approximations are typically implemented. This will be illustrated by the heat exchanger case study.

The third-order approximation can be expressed by:

$$\dot{z} = [\Gamma + D(z,\mu)]z + \Psi\mu + f^{(2)}(z) + g^{(1)}(z)\mu + f^{(3)}(z) + g^{(2)}(z)\mu \quad (9)$$

In this case, the number of parameters to be determined is $N_p = n[n+m+n'+nm+(n'+(n-1)' + (n-2)' + . . + 1) + n'm]$, where the "prime" operation is defined as $I' = I + (I-1) + (I+2) + . . + 1$. Consequently, the minimum number of steady states to be considered is also equal to $[n+n'+nn'(n'-n)]$ for the calculation of Γ, $f^{(2)}$ and $f^{(3)}$, and $[m+nm+n'm]$ for the calculation of Ψ, $g^{(1)}$ and $g^{(2)}$, respectively. For a 2×2 system with second- third- and fourth-order expansions N_p equals 22, 42 and 68, respectively. However, most state/manipulated variables do not appear or do not exhibit high-order components in all equations, thus N_p is drastically reduced by the elimination of unnecessary parameters. Furthermore, the coefficients of higher-order models are found numerically, and this procedure is easily automated. This will be illustrated by the case study later.

Controller Formulation and Stability Issues

With the well-defined structure of the resulting model, the polynomial expansion motivates the nonlinear stability analysis developed by Khambanonda et al. (1990). They use conic sectors to quantify magnitudes of nonlinear operators and develop a modified circle criterion for nonlinear systems. This development is greatly facilitated if the nonlinear operator is decomposed into a linear dynamic and a nonlinear static part. We will follow the path of transformations outlined in Figure 1 and first consider the case where the plant is nonlinear and the controller is linear. This will later be extended to the case where both the plant and the controller are allowed to be nonlinear.

Figure 1a depicts the closed-loop system containing the nonlinear plant NP and the linear controller LC. Using the polynomial approximation methodology described in the previous section, the nonlinear plant is replaced by an nth-order polynomial expansion that leads to the configuration in Figure 1b, where the uncertainty is accounted for as a linear perturbation term D. For the stability analysis, the nonlinear term has to be decomposed into a linear dynamic and a nonlinear static part, and this is achieved by using an input transformation suggested by Krener (1984). It must be noted that this transformation is only possible when the states (or the outputs) are in the range space of the inputs. This condition could be quite restrictive and would be satisfied if the nonlinear system meets the involutivity conditions discussed in detail by Krener (1984). After the decomposition, the linear operator LP becomes the first-order linearized form of NP (see Figure 1c):

$$\dot{x} = Ax + Bu \quad (10)$$

Details of this transformation can be found in Khambanonda et al. (1990). Finally, the linear part is placed in the frequency domain context, utilizing an additive uncertainty description:

$$LC\cdot LP = L + \Delta \quad (11)$$

with L defined as

$$L = [sI - A(d_i = 0)]^{-1} B\ LC$$

$$\Delta\epsilon\Delta \equiv \{\Delta\epsilon C^{n\times m}: \Delta = [(sI - A(d_i))^{-1} B\ LC],\ d_i^l \le d_i \le d_i^u\}$$

and after some block manipulations, one obtains the closed-loop configuration in Figure 1d.

Up to now, we focused on the nonlinear plant block transformation with the controller block containing an arbitrary linear controller. The simplest situation is when a proportional feedback law is employed, and this case is studied by Khambanonda et al. (1990), where no additional complexity exists in the form of dynamic controller elements or nonlinear compensation. Now, we shall generalize this to the case where any type of feedback compensation can be used and introduce the general configuration that will allow the application of the stability analysis. Figure 2a depicts the nonlinear plant (NP) together with a controller block that consists of both linear (LC) and nonlinear (NC) elements. The linear block (LC) can be obtained through any linear controller design methodology. The plant equations are again approximated using the polynomial expansion. Through block manipulation, one

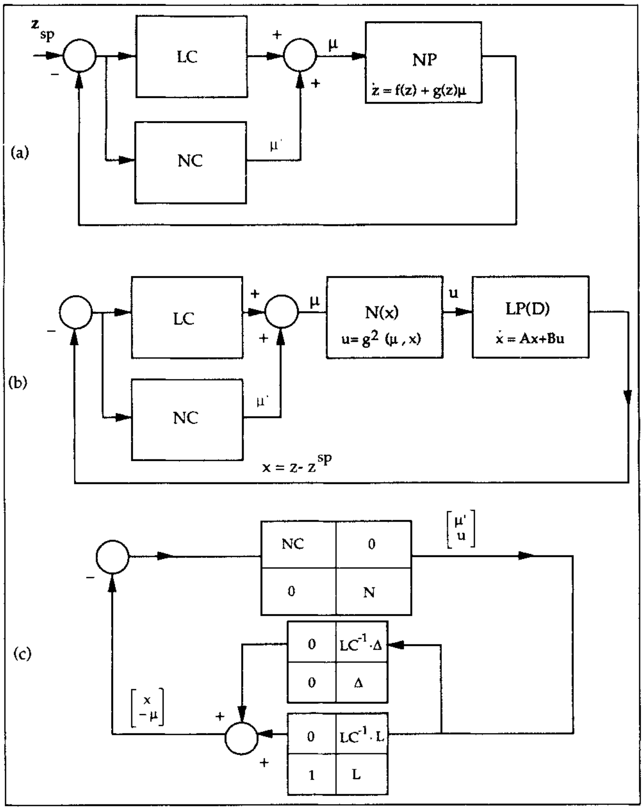

Figure 2. Evolution of the nonlinear transformation with combined linear-nonlinear controller.

can reduce the feedback scheme in Figure 2a to the block form in Figure 2b, where the primary operation is the decomposition of the nonlinear plant into its dynamic linear and static nonlinear parts as described before. With further block manipulations, one obtains the configuration in Figure 2c, and one can easily see that this representation is equivalent to that of Figure 1d. The difference lies in the subsystem structure which now has a block structure that shows the contribution of each element in the loop.

The stability of this feedback system with an arbitrary controller is tested by using an index of stability. The closed-loop system of Figure 1d is stable if

$$\text{index}(x) = \max_{\alpha} \frac{2\|N(x)\ \frac{1}{2}\ (\beta(\alpha)+\alpha)Ix\|_2}{(\beta(\alpha)-\alpha)\ \|x\|_2} < 1 \qquad (12)$$

where α and $\beta(\alpha)$ are sector bounds as defined by Khambanonda et al. (1990). This is a sufficient condition that helps us define guaranteed stable operating regions. Note that the development of the Index depends critically on the transformations performed where the nonlinear operator is decomposed into a linear dynamic part and a static nonlinear part. The polynomial model makes it possible to decompose the operator and this decomposition, in turn, allows the use of sector conditions to establish the stability criterion.

Since the states and the outputs are assumed to be bounded within the desired operating region, we can expect intuitively that these stability results will guarantee that $z(t)$ and $\mu(t)$ will lie within this region as long as $z(0)$ is sufficiently close to the origin.

Linear controllers

Once an approximate model is obtained, one can formulate a linear controller that can be tested using the index. The regions of stable operation thus obtained will provide the designer with the extent to which linear controllers can be used reliably. If conventional *PI*-type controllers are used, one will also be assured of asymptotic tracking, thus some performance specifications can also be imposed on the system. The *P/PI*-type controllers are industry standards, and by providing a framework to evaluate their potential for nonlinear systems is critical. Since the controller parameters (gains and reset times) have intuitive interpretation in terms of process response characteristics, one can also observe the trade-offs involved in constructing large stable operating regions versus satisfactory feedback performance. We shall illustrate this by the case study.

Nonlinear controller formulation

The transformation approach also allows us to choose a nonlinear controller consisting of a linear controller plus a correction term that takes into account the higher-order nonlinearities according to the order of the approximation used. The purpose of this controller is to make nonlinear systems follow an assigned linear-type behavior by compensating for the nonlinearity of the process with the higher-order terms of the control action. Here, we will follow the procedure proposed by Krener et al. (1987) by defining new state variables in the form:

$$z - z^0 = x + \phi^{(2)}(x). \qquad (13)$$

While inspired by Krener et al. (1987), we have to underline the fact that this controller has a fundamental difference since it is based on a polynomial model valid over the "desired operating region" as opposed to being valid at the point of expansion. The system of Eq. 6 can be written as:

$$\dot x (A+D)x + B[\mu + \nu^{(2)}(x) + \kappa^{(1)}(x)\mu] \qquad (14)$$

where

$\nu = \{\nu_{ijk} x_j x_k\}$; $\kappa = \{\kappa_{ijl} x_j\}$; $i,j = 1, \ldots n$; $k = j, \ldots n$; $l = 1, \ldots m$

The coefficients ν and κ can be calculated by solving the following problem:

$$f^{(2)}(x) - [A \cdot x, \phi^{(2)}(x)] = B \cdot \nu^{(2)}(x)$$
$$g^{(1)}(x) - [B \cdot \mu, \phi^{(2)}(x)] = B \cdot \kappa^{(1)}(x) \cdot \mu \qquad (15)$$

where $[a,b] = [\partial a(x)]/(\partial x^T) \cdot b(x) - [\partial b(x)]/(\partial x^T) \cdot a(x)$ is the Lie bracket of vectors **a** and **b**. Equation 15 represents a system of $n^2(n+1)/2 + mn^2$ *linear* equations in $n^2(n+1)/2 + mn(n+1)/2 + m^2 n$ unknown coefficients. This system of equations is generally overdetermined, and its solution is thus

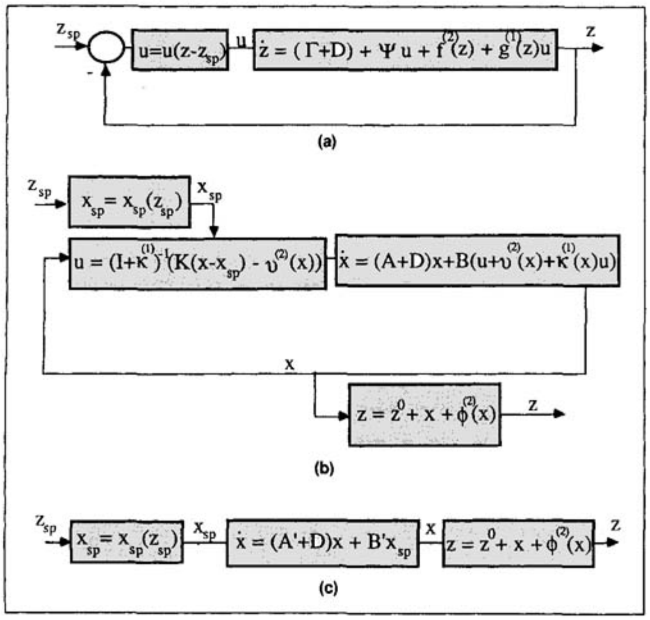

Figure 3. Evolution of the feedback system with nonlinear controller of Eq. 20.

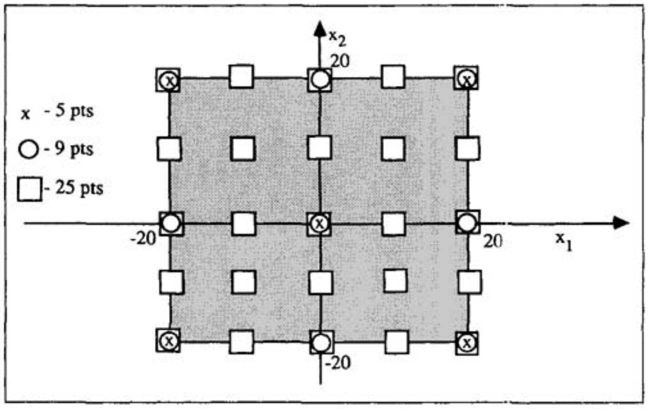

Figure 4. Selected steady states for p = 5, 9 and 25.

nonunique. One resorts to approximate solutions using a least square type approach (Karahan, 1989).

The feedback law can be chosen as:

$$\mu + \nu^{(2)}(x) + \kappa^{(1)}(x)\mu = K(x - x_{sp}) \tag{16}$$

or

$$\mu = [I + \kappa^{(1)}(x)]^{-1}[K(x - x_{sp}) - \nu^{(2)}(x)] \tag{17}$$

It is noted that the first-order coefficients of the control law in Eq. 17 is the linear control policy. The higher-order terms can be visualized as the correction terms for the process nonlinearity. Consequently, Eq. 17 yields the linear closed-loop response, i.e.

$$\dot{x} = (A' + D)x + B'x_{sp} \tag{18}$$

where $A' = A + BK$, and $B' = -BK$. With this controller formulation, the previous block configuration reduces to that of Figure 3 that shows the specific controller formulation and the resulting blocks. Here, the linear part can be utilized as a pole placement controller, while the nonlinear part is fixed by the system parameters. The design of the linear part is crucial and will be illustrated by the following examples.

Table 1. Model Parameters for Eq. 19

U_h = 198.8 W/km^2	A = 1.9 m^2	T_{pi} = 366.7 K
C_{pp} = 1.59 kJ/kg·K	M_p = 6.8 kg	T_{ci} = 295 K
C_{pc} = 1.91 kJ/kg·K	M_c = 18.15 kg	F_p = 0.19 kg/s
F_c = 0.315 kg/s	T^0_{co} = 317.84 K	T^0_{po} = 321.22 K

Case Study of a Heat-Exchanger

Here we will study the same example previously used by Alsop and Edgar (1989) and Khambanonda et al. (1990) and show the model approximation alternatives and the impact of various controller design strategies. The model equations are given below

$$\frac{dT_{co}}{dt} = \frac{2}{M_c}\left[F_c(T_{ci} - T_{co}) - \frac{U_h A_h}{C_{pc}}\Delta T_{lm}\right]$$

$$\frac{dT_{po}}{dt} = \frac{2}{M_p}\left[F_p(T_{pi} - T_{po}) + \frac{U_h A_h}{C_{pp}}\Delta T_{lm}\right]$$

$$\Delta T_{lm} = \frac{[(T_{ci} - T_{po}) - (T_{co} - T_{pi})]}{\ln[(T_{ci} - T_{po})/(T_{co} - T_{pi})]} \tag{19}$$

with the model parameters given in Table 1. The model equations are expanded according to Eq. 6. The comparison between the model equations and Eq. 6 reduces N_p from 22 to 14 (Table 2) and the corresponding minimum value of p is 5. The operating region within which the system stability is tested is chosen such that the maximum temperature variation is within ±20.0 K from the nominal operating point. The five steady states required for the parameter fit are chosen as shown in Figure 4. These steady states express the dominant characteristics of the system in all extreme cases. The process characteristics not captured by these points are accounted for by $D(z,\mu)$ whose bounds are depicted in Table 3.

Table 2. Model Equations vs. Eq. 6

γ_{11} = 8.4459	β_{111} = -1.324×10^{-2}	β_{212} = 3.767×10^{-4}	φ_{111} = -0.108
γ_{12} = -8.3580	β_{112} = -1.175×10^{-4}	β_{222} = -4.197×10^{-2}	φ_{222} = -0.294
γ_{21} = -27.0801	β_{122} = 1.309×10^{-2}	Ψ_{11} = 32.507	
γ_{22} = 26.7982	β_{211} = 4.245×10^{-2}	Ψ_{22} = 107.853	

Table 3. Bounds for $D(z,\mu)$

No. of Points	d_1^l	d_1^u	d_2^l	d_2^u
5	-1.59×10^{-2}	1.75×10^{-2}	-5.73×10^{-2}	5.21×10^{-2}
9	-7.44×10^{-5}	3.95×10^{-4}	-1.29×10^{-3}	2.43×10^{-4}
25	-5.94×10^{-5}	3.92×10^{-4}	-1.28×10^{-3}	1.95×10^{-4}
49	-5.94×10^{-5}	3.92×10^{-4}	-1.28×10^{-3}	1.94×10^{-4}

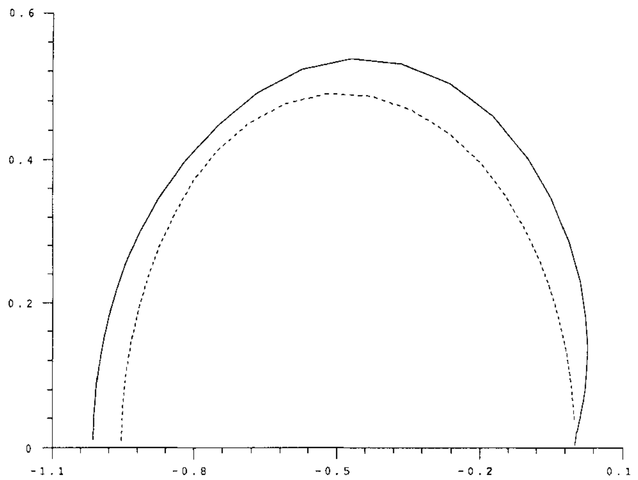

Figure 5. Characteristic loci of the linearized model.

Linear controller case

First for the case with $p=5$, we will use the proportional controller K in the form:

$$K=k\begin{bmatrix}0.17 & 0\\ 0 & -0.0072\end{bmatrix} \tag{20}$$

where k is an adjustable parameter such that if $k=1.0$, the characteristic loci of the linearized model crosses the critical point, indicating instability as shown in Figure 5.

The closed loop system is arranged in the configuration shown in Figure 2 by setting $LC=K$ and thus, Eq. 11 yields L and Δ. The index (x_1,x_2) is computed and the curve along which index = 1 is plotted in Figure 6 as a function of k to locate the regions of stable operation. One can illustrate the implications of these plots by simulations that utilize the original nonlinear model. The simulations are carried out with $k=0.18$ for various set point changes (Figure 7). Stability analysis (Figure 6) indicates that set point changes such that x_1 is negative and x_2 is positive (fourth quadrant in the state space) will yield stable responses while set point changes in the second

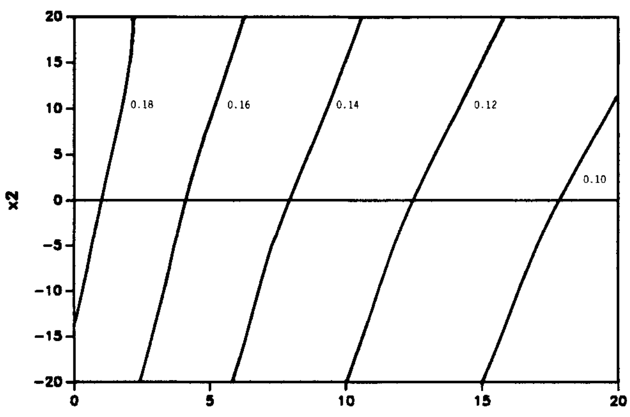

Figure 6. Stability regions obtained from the second-order expansion with proportional controller and $p=5$.

Only first and fourth quadrants are shown.

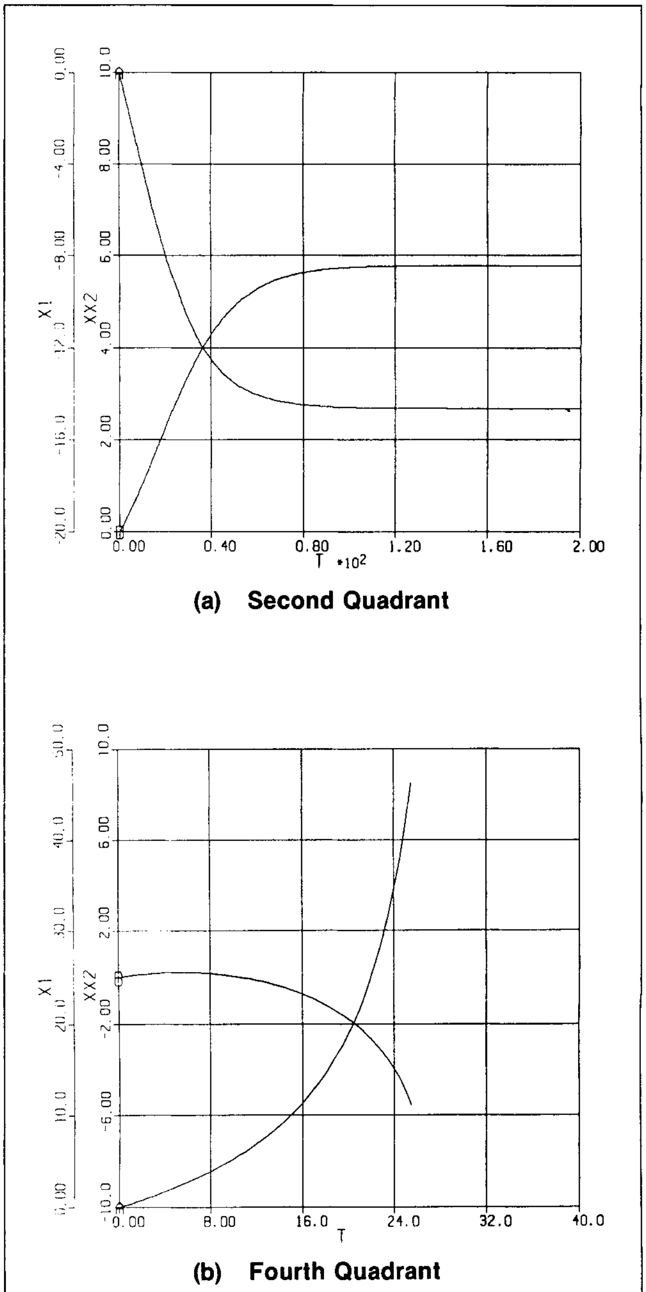

Figure 7. Closed-loop responses to set-point changes for proportional controller.

a: $k=0.18$, and $x_1^{sp}=-5.0$, $x_2^{sp}=5.0$.
b: $k=0.18$, and $x_1^{sp}=5.0$, $x_2^{sp}=-5.0$.

Table 4. Parameters Values

No. of Point	γ_{11}	γ_{12}	β_{111}	β_{112}	β_{122}
5	8.446	-8.358	-1.324×10^{-2}	-1.175×10^{-4}	1.309×10^{-2}
9	-4.03×10^{-2}	3.92×10^{-2}	3.06×10^{-5}	3.75×10^{-5}	-5.73×10^{-5}
25	-4.18×10^{-2}	4.08×10^{-2}	3.04×10^{-5}	4.23×10^{-5}	-6.23×10^{-5}
49	-4.14×10^{-2}	4.05×10^{-2}	2.95×10^{-5}	4.27×10^{-5}	-6.21×10^{-5}

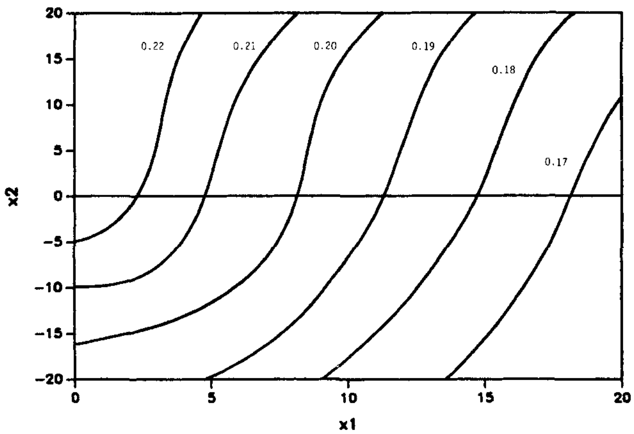

Figure 8. Stability regions obtained from the second-order expansion with proportional controller and $p=9$.

Only first and fourth quadrants are shown.

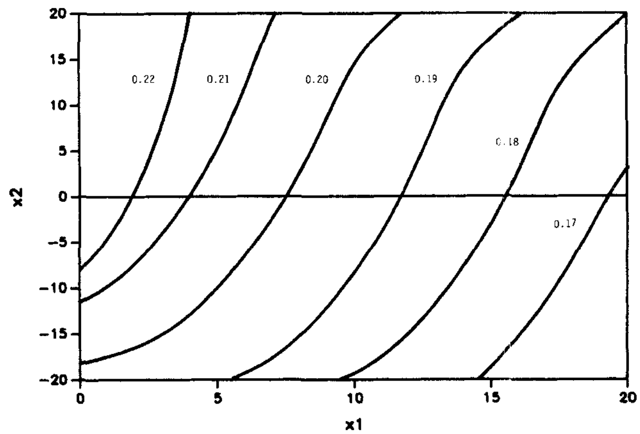

Figure 10. Stability regions obtained from the third-order expansion with proportional controller and $p=9$.

Only first and fourth quadrants are shown.

quadrant will most likely be unstable (x_1 is positive and x_2 is negative). This is demonstrated by the simulations in Figures 7a and 7b respectively. Note that the directionality is captured by this nonlinear stability analysis where the linear tests would fail to do so.

The above case is repeated with the values of the coefficients modified by setting $p=9$, 25 and 49 to generate new polynomial expansions. The steady states chosen (Figure 4) are assumed to be equally weighed except at the nominal point where the expansion is made exact. As a result of the better fitting, the uncertainties due to higher-order terms are reduced by increasing p. However, when $p\geq9$, minimum improvement is achieved indicating that the full model is sufficiently represented by nine steady states. In addition, the values of the parameters listed in Table 4 show substantial changes from $p=5$ to $p=9$ while slight changes observed thereafter, yielding the same conclusion. The expanding stable operating regions with the increase in p (Figures 8 and 9) also indicate that more accurate results can be achieved with this scheme. Again, the stable regions are very similar for $p=9$ and $p=25$.

Next the third order expansion is tested by using the minimum number of steady states, i.e., $p=9$. Since $g^{(1)}(z)$ is exactly equal to $g(x)$, the term $g^{(2)}(x)$ in Eq. 14 can be omitted. This

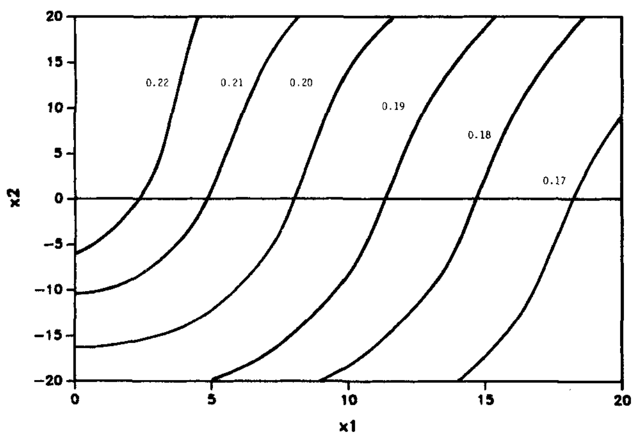

Figure 9. Stability regions obtained from the second-order expansion with proportional controller and $p=25$.

Only first and fourth quadrants are shown.

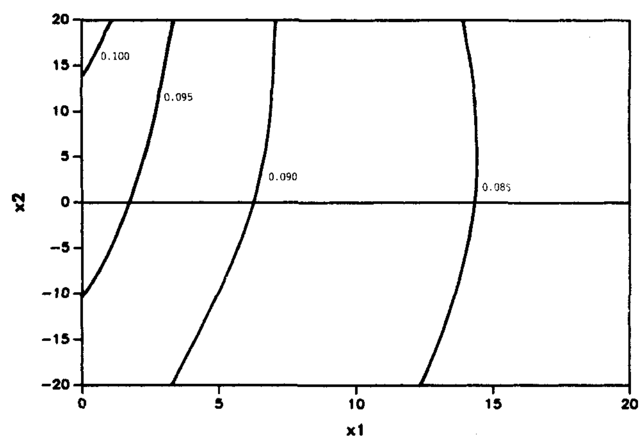

Figure 11. Stability regions obtained from the second-order expansion with PI controller and $p=9$.

Only first and fourth quadrants are shown.

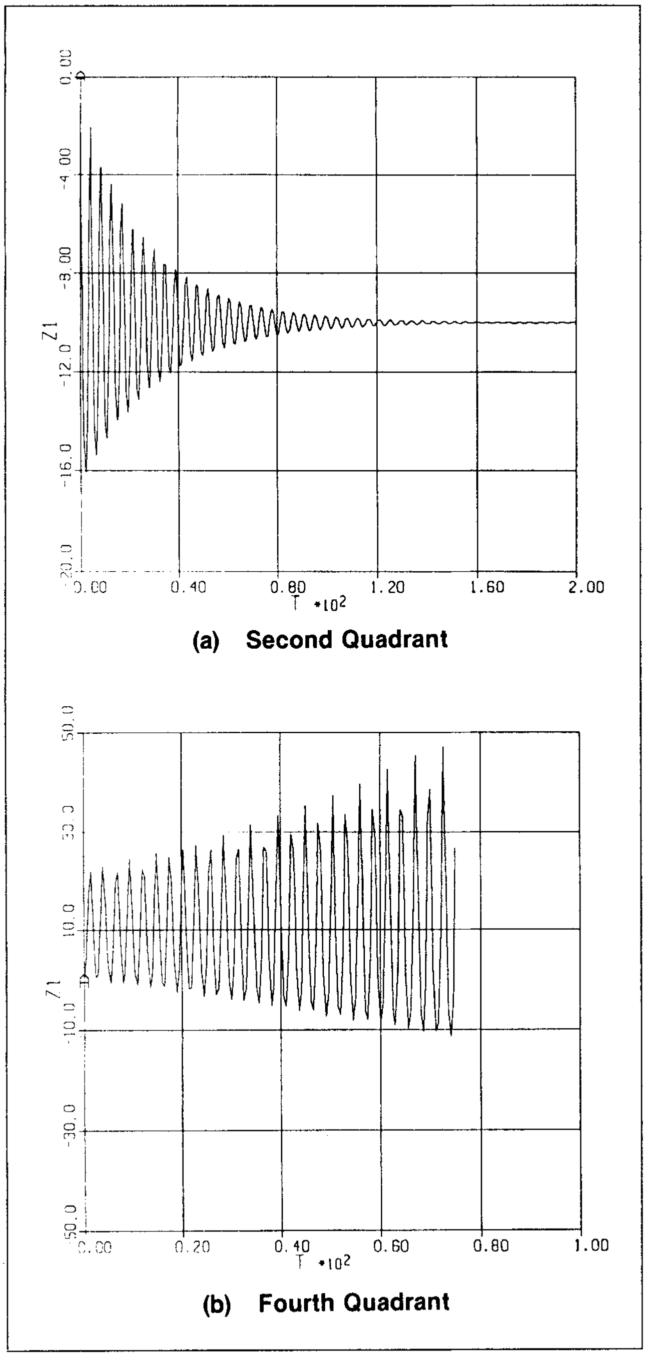

Figure 12. Closed-loop response to set-point changes for PI controller.

a: $k = 0.15$, and $x_1^{sp} = -10.0$, $x_2^{sp} = 10.0$.
b: $k = 0.15$, and $x_1^{sp} = 10.0$, $x_2^{sp} = -10.0$.

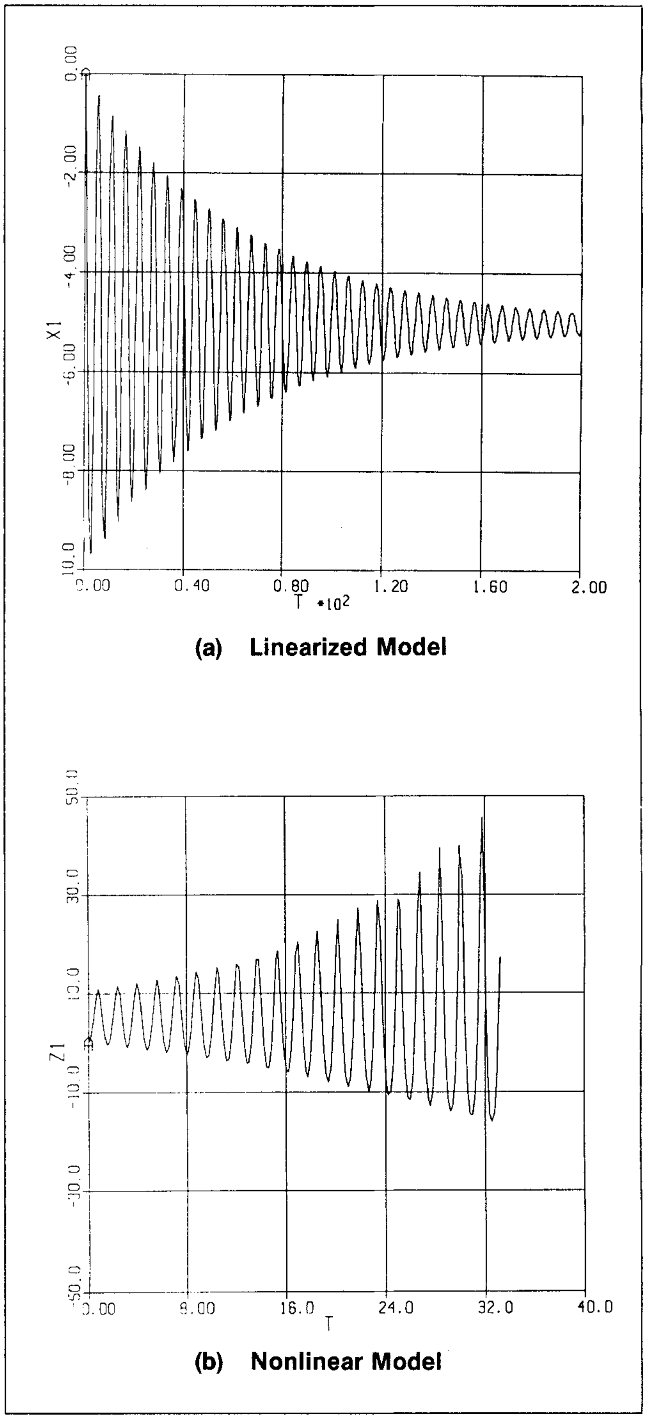

Figure 13. Closed-loop response to set-point changes for PI controller with $k = 0.5$.

a: $x_1^{sp} = 5.0$, $x_2^{sp} = 5.0$
b: $x_1^{sp} = 5.0$, $x_2^{sp} = 5.0$

reduction yields $N_p = 22$ and the minimum value of $p = 9$. Again, the stability regions as a function of k are calculated and shown in Figure 10.

The similarity in the size and shape of the stability regions generated by the 2nd-order expansion and the 3rd-order expansion with nine steady-state points implies that the second-order expansion sufficiently describes the behavior of this process. However, this cannot be generalized and will depend on the type and the severity of the nonlinearities involved. Usually, higher-order expansions lead to more accurate models, but along with the fact that the number of parameters grows exponentially, the sector bound calculations also become more conservative especially if the domain Z is large. In some instances, however, when an excessively higher-order expansion is used to imitate mild nonlinear behavior, adverse effects result

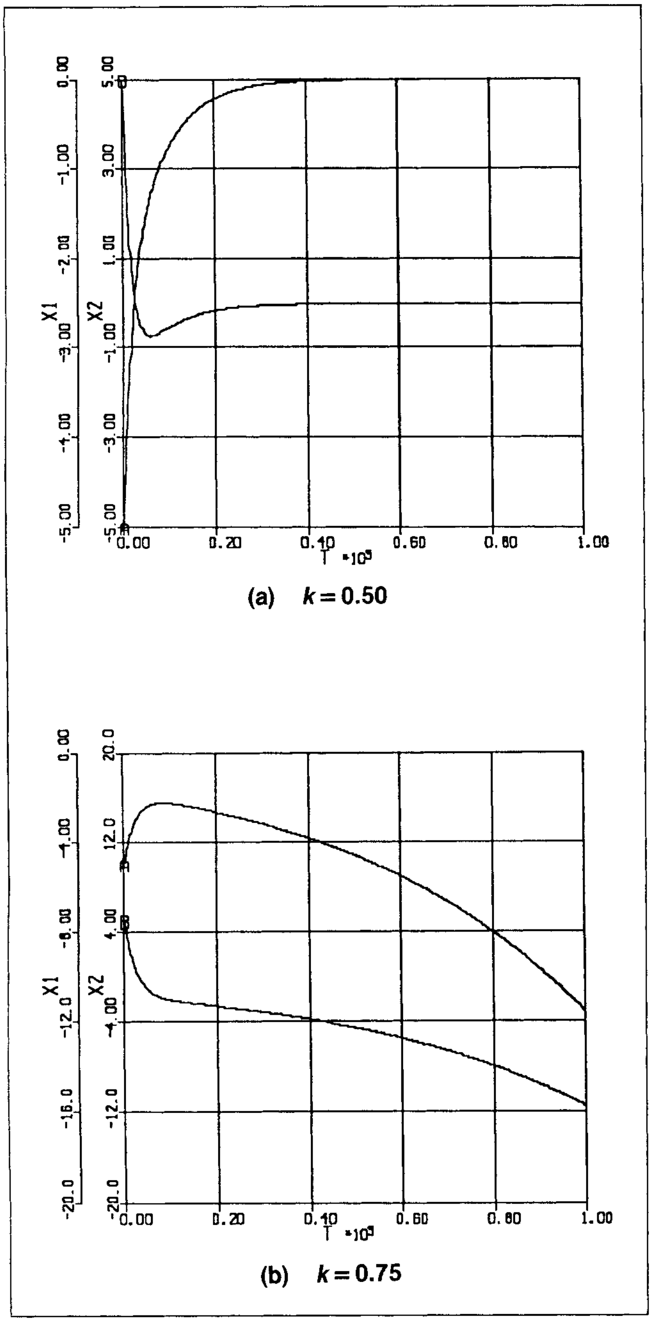

Figure 14. Closed-loop response of nonlinear controller.

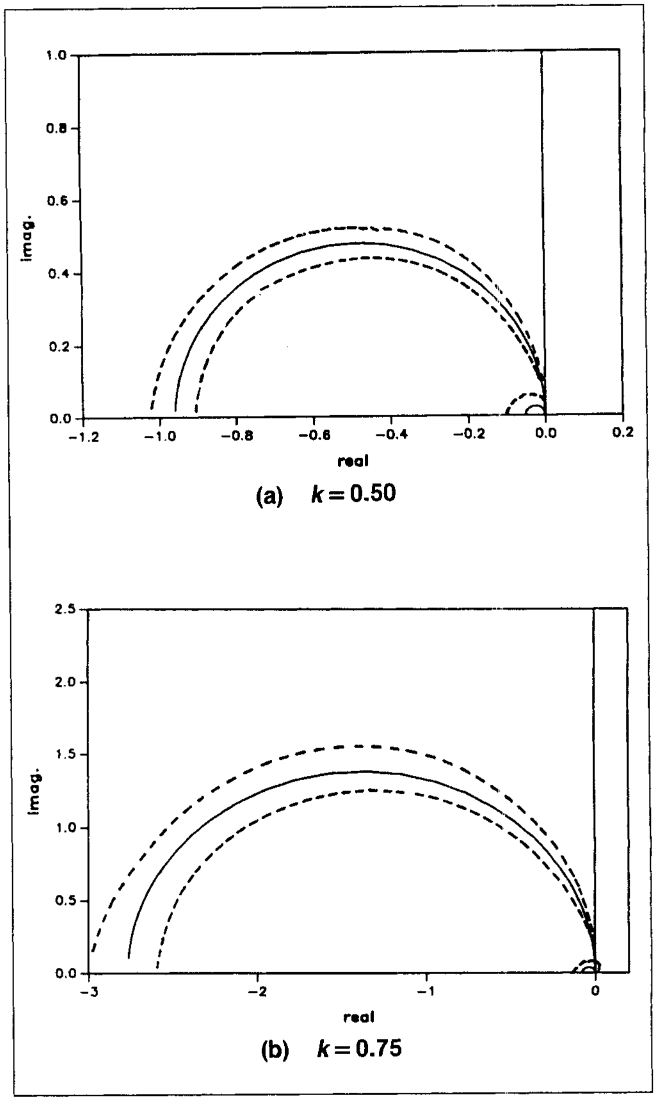

Figure 15. Characteristic loci plots for the system.

due to excessive degree of freedom for the parameter fitting procedure. While the expansion performs satisfactorily at all the steady states selected, transient behavior of this expansion may deviate dramatically from the original model, resulting in large uncertainty bounds and hence potentially conservative predictions. For these reasons, the order of expansion plays an important role and should be carefully selected. In general, one can roughly choose this number by inspecting the form of nonlinear terms the original model contains. For example, the reaction terms with small activation energy requires higher-order expansions while the square-root terms may be satisfied with low order ones.

To impose asymptotic tracking of the states, we can add integral action to the linear controller. For simplicity of exposition, the PI-controller for this case study is assumed to take the form:

$$K = k\begin{bmatrix} 0.1\left(1.0 - \dfrac{1.0}{0.01\,s}\right) & 0 \\ 0 & -0.005\left(1.0 - \dfrac{1.0}{0.01\,s}\right) \end{bmatrix} \tag{21}$$

The parameters of the second-order expansion are fitted with $p = 5$ and the stability region of this feedback system is constructed. It is observed that the integral action, while it eliminates the steady-state offset, leads to a degradation in stability margins as reflected by a larger sector of the linear part in Figure 1. This is also depicted by a smaller stability region observed in Figure 11. The stability region is tested by simulations (Figure 12) where the closed-loop system is shown to be unstable with $k = 0.15$. This hints at a potential conservatism of the stability analysis that can be attributed to two factors.

The first is the inherent conservatism associated with the sector condition especially with the added integral action. The second factor is that the regions in Figure 11 guarantee system stability for all frequencies while the simulations are made at a particular range of frequencies. The results, however, predict a more accurate description of the system stability than that would be obtained from traditional linear analyses. Figure 13 shows a simulation using the linear model and the nonlinear model for $k=0.5$ to stress the inability of the linear model to capture the instability of the nonlinear system when the system deviates from the point of linearization.

Nonlinear controller

The controller of Eq. 17 is implemented resulting in the linear system of Eq. 18. Without a disturbance, the system becomes a feedforward process with an uncertainty where the response is controlled by choosing K and varying the set point. In Figure 14, several simulations are shown as a function of k where K is described by Eq. 20. The gains are significantly higher than that would be allowed by the linear part of the controller only.

The system stability can be assessed by a Nyquist plot of the system eigenvalues with the uncertainty bands as shown in Figure 15. The perturbed eigenvalues are contained within the following bounds (Wilkinson, 1965):

$$|\lambda_i(L)-\lambda_i(L+\Delta)| \leq \gamma(L)\cdot\delta \tag{22}$$

where $\gamma(L)$ is the condition number of the matrix L defined as:

$$\gamma(L)=\frac{\sigma_{\max}(L)}{\sigma_{\min}(L)} \tag{23}$$

and

$$\delta=\max_{d_i^l\leq d_i\leq d_i^u}\sigma_{\max}(\Delta) \tag{24}$$

This reflects an unstructured uncertainty description and is sufficient for the purposes of demonstration here. However, one can improve on this significantly by the use of other uncertainty quantification strategies (Palazoglu and Khambanonda, 1989). This aspect will not be explored in this article.

Note that this is now a linear stability test with uncertainty since the nonlinear controller produces an exact cancellation of the second-order terms. As Figure 3 shows, simultaneous use of the polynomial transformation and the nonlinear controller effectively yields a linear open-loop system with uncertainty.

Conclusions

A transformation approach is presented to approximate nonlinear systems. The method involves polynomial expansions with a linear bounded perturbation representing the contribution of higher-order terms and other possible sources of uncertainty. The effects of parameter fitting techniques and the expansion order on the model accuracy is demonstrated, resulting in substantial improvements. The generalized model based on this expansion is used to test the stability of the nonlinear feedback system through the use of the sector bound condition yielding stability regions on the state plane. Linear controllers are shown to be easily tunable and thus interpretable for nonlinear systems. Furthermore, a nonlinear controller design strategy is employed to account for higher-order terms and thus allowing higher gains to be used for a better closed-loop response.

Acknowledgment

The authors would like to acknowledge the anonymous reviewers for their many valuable and in-depth comments that helped considerably in the revision of the manuscript.

Notation

A_h = heat exchanger area
C_p = heat capacity
D = uncertainty matrix
F = flowrate
L = linear element
LC = linear controller
m = number of manipulated variables in the system
M = heat-exchanger holdup
n = number of state variables in the system
N = nonlinear element
NC = nonlinear controller
N_p = number of parameters required for the higher-order expansion
NP = nonlinear plant
k = adjustable parameter for controllers
p = number of selected steady states
s = Laplace transform variable
t = time
T = temperature
u = vector of reduced manipulated variables
U = domain of vector u
U_h = heat transfer coefficient
w = vector of weight functions
v = vector of weight functions
x = vector of reduced state variables
z = vector of state variables
Z = domain of z

Greek letters

α = lower sector bound
β = upper sector bound
Δ = uncertainty block in frequency domain
ΔT_{lm} = log-mean temperature
μ = vector of manipulated variables
μ' = output vector from nonlinear control element (NC)

Superscripts

0 = nominal steady-state
(*) = of order *
l = lower bound
s = steady state
T = transpose operation
u = upper bound

Subscripts

c = cold
ci = cold inlet
co = cold outlet
h,i,j,k,l = indices
p = hot
pi = hot inlet
po = hot outlet

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Manuscript received May 9, 1990, and revision received May 6, 1991.